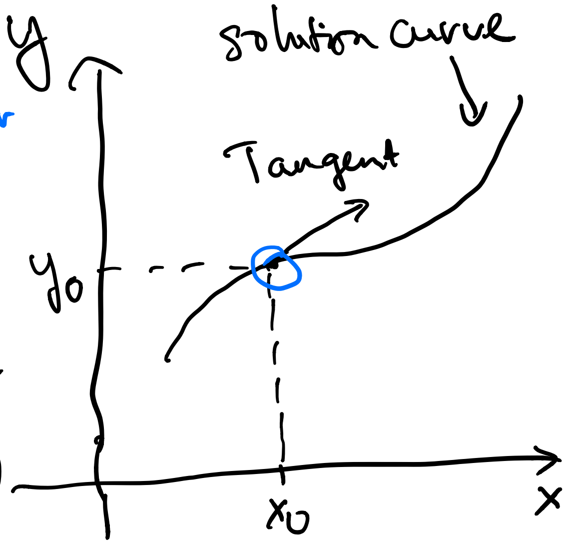
- Solution Curves

$\boxed{y' = f(x,y)}$ ⇐ 1st order

From calculus

the slope of the tangent line is exactly $y' = f(x,y)$



⇒ (Doing it systematically) Slope field

- Autonomous Equations

Def: $y' = f(y)$

Goal: Behavior of solutions without actually finding those solutions

⊕ Critical points = fixed points = Nodes = Equilibrium points. = points where $f(y) = 0$.

- Represent solutions that are constant.

Ex: $y' = \underbrace{y^2 - 1}_{f(y)}$

$y' = 0$

$y^2 - 1 = 1^2 - 1 = 0$

$f(y) = y^2 - 1 = 0 \iff y = \pm 1$

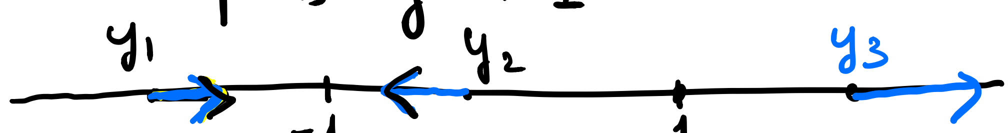
Function $y = 1$ is a solution
$y = -1$

- Classification of critical points. (x_0, y_0)

  - Asymptotically stable (sink): If solutions that begin near y_0 approach y_0 as $x \uparrow$.
  - Unstable / source: If solutions that begin near y_0 move away from y_0 as $x \uparrow$
  - Semi-stable: neither a sink or a source (stable on one side and unstable on another side)

<u>Ex</u>: $y' = y^2 - 1$

Critical points $y = \pm 1$



$\Rightarrow$ stable $\quad\quad$ $\Rightarrow$ un-stable.

phase line

$y_1 < -1$, $y' = y^2 - 1$, at y_1, $y_1^2 - 1 > 0$

$y \uparrow$ towards -1

$-1 < y_2 < 1$, $y' = y^2 - 1$ at y_2, $y_2^2 - 1 < 0$

$y \downarrow$ towards -1

-1 : stable.

1: not stable $\Rightarrow$ source.

$y_3 > 1$, $y' = y^2 - 1$ at y_3, $y_3^2 - 1 > 0$

$y \uparrow$ away from 1.

Summary: Phase line

- $f(y) = 0 \Rightarrow$ critical points
- Check for sign of $f(y)$ in each interval.

  ($f(y) > 0$, $y \uparrow$

  $f(y) < 0$, $y \downarrow$)
- Mark arrows on intervals and read out classification.

2.2. Separable Equations

Def: 1st order ODE of the form

$$\frac{dy}{dx} = \underbrace{g(x)\,h(y)}$$

Keyword: Separable.

Solving a separable ODE:

+ Separate x away from y.

$$\frac{dy}{dx} = g(x)\,h(y)$$

$$\Leftrightarrow \quad \frac{dy}{h(y)} = g(x)\,dx$$

+ Integrate both sides

$$\int \frac{dy}{h(y)} = \int g(x)\,dx + C$$

+ Evaluate indefinite integrals $\Rightarrow$ Implicit solution.

Ex: $y' = \frac{dy}{dx} = \frac{y}{1+x} = y \cdot \frac{1}{1+x}$

① $\frac{dy}{y} = \frac{dx}{1+x}$

② $\int \frac{dy}{y} = \int \frac{dx}{1+x}$

$$\boxed{\ln(y) = \ln(1+x) + C}$$

$\Leftarrow$ implicit general solution

Implicit: Relation of y and x

Explicit solution: y written as a function of x

$$e^{\ln(y)} = e^{\ln(1+x)+C}$$

$$\boxed{y = e^{C}(1+x)}$$ ⇐ explicit solution

Ex: $y' = (x+1)\cos(x^2+2x)$

$$\frac{dy}{dx} = (x+1)\cos(x^2+2x)$$

$$\int dy = \int (x+1)\cos(x^2+2x)\,dx$$

$$y =$$

(u-sub: $u = x^2+2x$, $du = (2x+2)\,dx$

$$\int (x+1)\cos(x^2+2x)\,dx = \int \cos u \,\frac{du}{2}$$

$$= \frac{1}{2}\sin u + C \,)$$

$$\boxed{y = \frac{1}{2}\sin(x^2+2x) + C}$$ ⇐ general solution.

Ex (IVP): $y' = (x+1)\cos(x^2+2x)$, $y(0) = 2020$.

- Step 1: Find the general sol
- Step 2: Plug in numbers (initial conditions) to determine constants

General solution:

$$y = \frac{1}{2}\sin(x^2+2x) + C.$$

$$y(0) = 2020 \Rightarrow 2020 = \frac{1}{2}\sin(0^2+2.0)+C$$

$$2020 = \frac{1}{2}\sin 0 + C = C$$

Solution to the IVP:

$$y = \frac{1}{2}\sin(x^2+2x) + 2020.$$